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ESTIMATION IN MIXTURES OF POISSON AND MIXTURES OF EXPONENTIAL DISTRIBUTIONS

by A. CLIFFORD COHEN, JR.
Aero-Astrodynamics Laboratory

NASA

George C. Marshall Space Flight Center, Huntsville, Alabama

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A. Clifford Cohen, Jr.*

George C. Marshall Space Flight Center

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ABSTRACT

In the analysis of experimental data, many of the distributions encountered are the result of combining two or more separate component distributions. Estimation in these compound or mixed distributions is therefore of particular interest to aerospace scientists. Estimators are derived for the parameters of a compound Poisson distribution with probability density function

$$f(x) = \alpha \frac{e^{-\mu} \mu^x}{x!} + (1 - \alpha) \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0,1,2,\ldots$$

and for a compound exponential distribution with probability density function

$$f(x) = \alpha(1/\mu)e^{-x/\mu} + (1 - \alpha) (1/\lambda)e^{-x/\lambda}, \quad x \geqq 0$$

where α is the proportionality factor ($0 \leqq \alpha \leqq 1$) and where μ and λ are component parameters. In addition to the more general case in which all parameters must be estimated from sample data, several special cases are considered in which one or more of the parameters are known in advance of sampling.

*Professor of Mathematics, University of Georgia, Athens, Georgia. The research reported in this paper was performed under NASA Contract NAS8-11175 with the Aerospace Environment Office, Aero-Astrodynamics Laboratory, Marshall Space Flight Center, Huntsville, Alabama. Mr. O. E. Smith and Mr. J. D. Lifsey are the NASA contract monitors.

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ESTIMATION IN MIXTURES OF POISSON AND MIXTURES OF EXPONENTIAL DISTRIBUTIONS

By

A. Clifford Cohen, Jr.

TERRESTRIAL ENVIRONMENT GROUP
AEROSPACE ENVIRONMENT OFFICE
AERO-ASTRODYNAMICS LABORATORY

FOREWORD

This report presents results of an investigation performed by the Department of Statistics, University of Georgia, Athens, Georgia, as a part of NASA Contract NAS8-11175 with the Aerospace Environment Office, Aero-Astrodynamics Laboratory, NASA-George C. Marshall Space Flight Center, Huntsville, Alabama. Dr. A. C. Cohen, Jr. was the principal investigator. The NASA contract monitors are Mr. O. E. Smith and Mr. J. D. Lifsey.

The results of this study represent a contribution in the area of statistical estimation from compound (mixed) frequency distributions. The methods presented are straightforward and may be easily adapted to practical application.

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SUMMARY

In the analysis of experimental data, many of the distributions encountered are the result of combining two or more separate component distributions. Estimation in these compound or mixed distributions is therefore of particular interest to aerospace scientists. Estimators are derived for the parameters of a compound Poisson distribution with probability density function

$$f(x) = \alpha \frac{e^{-\mu} \mu^{x}}{x!} + (1 - \alpha) \frac{e^{-\lambda} \lambda^{x}}{x!}, \quad x = 0,1,2,\ldots$$

and for a compound exponential distribution with probability density function

$$f(x) = \alpha(1/\mu)e^{-x/\mu} + (1 - \alpha) (1/\lambda)e^{-x/\lambda}, \quad x \geqq 0$$

where α is the proportionality factor ($0 \leqq \alpha \leqq 1$) and where μ and λ are component parameters. In addition to the more general case in which all parameters must be estimated from sample data, several special cases are considered in which one or more of the parameters are known in advance of sampling.

I. INTRODUCTION

Many of the distributions encountered in the analysis of experimental data are the result of combining two or more separate component distributions. Accordingly, estimation in these compound or mixed distributions is of particular interest to aerospace scientists. A previous paper [2] dealt with estimation in mixtures of two Poisson distributions; these previous results are extended here to include several special cases wherein one or more of the parameters of the compound Poisson distribution are known, and in addition analogous estimators are derived for the parameters of the compound exponential distribution.

The author wishes to acknowledge the assistance of Mr. Frank Clark for his work in establishing the IBM 7094 computer program described in Section IV and the Appendix.

II. MIXTURES OF TWO POISSON DISTRIBUTIONS

1. The Probability Density Function

The probability density function of a compound distribution composed of two Poisson components with parameters μ and λ, respectively, combined in proportions α and $1 - \alpha$ may be written as

$$f(x) = \alpha \frac{e^{-\mu} \mu^{x}}{x!} + (1 - \alpha) \frac{e^{-\lambda} \lambda^{x}}{x!} . \qquad \begin{cases} x = 0,1,2,\ldots \\ 0 \leqq \alpha \leqq 1 \end{cases} \tag{1}$$

For convenience and without any loss of generality, we assume $\mu > \lambda$.

2. Three-Moment Estimators

The following estimating equations result from equating the first three factorial moments of a sample of size n to the corresponding theoretical moments.

$$\left.\begin{aligned} \alpha &= \frac{(\bar{x} - \lambda)}{(\mu - \lambda)} \\ \bar{x}\theta - \Gamma &= \nu_{[2]} \\ \bar{x}(\theta^2 - \Gamma) - \Gamma\theta &= \nu_{[3]} \end{aligned}\right\}, \tag{2}$$

where

$$\theta = \mu + \lambda \qquad \text{and} \qquad \Gamma = \mu\lambda, \tag{3}$$

and where the sample factorial moment $\nu_{[k]}$ is given by

$$\nu_{[k]} = \sum_{x=0}^{R} x(x - 1) \ldots (x - k + 1) \frac{n_x}{n} , \tag{4}$$

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in which R is the largest observed (sample) value of the random variable x, n_x is the sample frequency of x, and

$$n = \sum_{x=0}^{R} n_x .$$

For simplicity of notation, $\bar{x}$ has been written in place of $\nu_{[1]}$ for the first sample factorial moment.

On solving the last two equations of (2) simultaneously for Γ and θ, it follows that

$$\left.\begin{aligned} \theta^* &= \frac{\nu_{[3]} - \bar{x}\,\nu_{[2]}}{\nu_{[2]} - \bar{x}^2} \\ \Gamma^* &= \bar{x}\theta^* - \nu_{[2]} \end{aligned}\right\}, \tag{5}$$

where the asterisk (*) distinguishes estimators from the parameters being estimated. The required estimators of μ and λ follow as

$$\left.\begin{aligned} \mu^* &= \frac{1}{2}\left[\theta^* + \sqrt{\theta^{*2} - 4\Gamma^*}\right] \\ \lambda^* &= \frac{1}{2}\left[\theta^* - \sqrt{\theta^{*2} - 4\Gamma^*}\right] \end{aligned}\right\}. \tag{6}$$

These estimators are the two roots r_1 and r_2 of the quadratic equation

$$Y^2 - \theta^* Y + \Gamma^* = 0, \tag{7}$$

where $\mu^* = r_1$ and $\lambda^* = r_2$, $(r_1 > r_2)$. The proportionality parameter α is estimated from the first equation of (2) as $\alpha^* = (\bar{x} - \lambda^*)/(\mu^* - \lambda^*)$.

The estimators given in equation (6) were originally derived by Rider [3], but he employed ordinary rather than factorial moments with the result that his derivations were somewhat complicated and his expressions for θ^* and Γ^* were more involved than those given here.

3. Estimators Based on the First Two Sample Moments and the Sample Zero-Frequency

It is well known that the higher sample moments are subject to appreciable sampling error, and in an effort to reduce errors from this source, the estimating equation based on the first two sample moments and the sample zero-frequency, was derived [1] as

$$\frac{\bar{x} - \lambda}{G(\lambda) - \lambda} = \frac{n_o/n - e^{-\lambda}}{e^{-G(\lambda)} - e^{-\lambda}}, \tag{8}$$

in which

$$G(\lambda) = \frac{\nu_{[2]} - \bar{x}\lambda}{\bar{x} - \lambda}, \tag{9}$$

where n_o is the sample zero-frequency. Equation (8) can be solved for λ^{**} using standard iterative procedures and, with λ^{**} thus determined, estimators of μ and α follow as

$$\left.\begin{aligned} \mu^{**} &= \frac{\nu_{[2]} - \bar{x}\,\lambda^{**}}{\bar{x} - \lambda^{**}} \\ \alpha^{**} &= \frac{\bar{x} - \lambda^{**}}{\mu^{**} - \lambda^{**}} \end{aligned}\right\}. \tag{10}$$

The double asterisk (**) distinguishes these estimators from the three-moment estimators and in turn from the parameters being estimated. Unfortunately, no simple procedure for solving equation (8) has been devised. However, a computer program based on iterative procedures described by Whittaker and Robinson [4, Chap. VI] has been developed (see Appendix) to solve equation (8), using as a first approximation the three-moment estimate of λ given by equation (6).

4. Estimation With Some Parameters Specified

a. α Known

In this case, we need only estimate μ and λ; for this purpose, the first two equations of (2) may be written as

$$\left.\begin{aligned} \alpha &= \frac{\bar{x} - \lambda}{\mu - \lambda} \\ \bar{x}(\mu + \lambda) - \mu\lambda &= \nu_{[2]} \end{aligned}\right\}, \tag{11}$$

where θ and Γ have been replaced by their defining relations as given in equation (3).

With α known, we obtain the following quadratic equation in λ from the two equations of (11):

$$\lambda^2 - 2\bar{x}\lambda + \frac{\bar{x}^2 - \alpha\nu_{[2]}}{1 - \alpha} = 0. \tag{12}$$

On solving equation (12)

$$\lambda^* = \bar{x} - \sqrt{\frac{\alpha(\nu_{[2]} - \bar{x}^2)}{1 - \alpha}}\ , \tag{13}$$

and from the first equation of (11)

$$\mu^* = [\bar{x} - \lambda^*(1 - \alpha)]/\alpha. \tag{14}$$

b. <u>α and μ Known</u>

In this case, λ may be estimated from the first equation of (11) as

$$\lambda^* = \frac{\bar{x} - \alpha\mu}{1 - \alpha} . \tag{15}$$

c. <u>α and λ Known</u>

In this case, it follows from equation (11) that

$$\mu^* = \frac{\bar{x} - (1 - \alpha)\lambda}{\alpha} . \tag{16}$$

d. <u>μ Known</u>

In this case, we may employ equations (11) to estimate α and λ. Accordingly, from the second equation of (11)

$$\lambda^* = \frac{\nu_{[2]} - \bar{x}\mu}{\bar{x} - \mu} , \tag{17}$$

and from the first equation of (11)

$$\alpha^* = \frac{\bar{x} - \lambda^*}{\mu - \lambda^*} . \tag{18}$$

e. <u>λ Known</u>

In this case, the second equation of (11) gives

$$\mu^* = \frac{\nu_{[2]} - \bar{x}\lambda}{\bar{x} - \lambda} , \tag{19}$$

and from the first equation of (11)

$$\alpha^* = \frac{\bar{x} - \lambda}{\mu^* - \lambda} \,. \tag{20}$$

III. MIXTURES OF TWO EXPONENTIAL DISTRIBUTIONS

1. The Probability Density Function

In many respects the exponential distribution may be thought of as a continuous analog to the discrete Poisson distribution. In any event, estimating equations in mixtures of two exponential distributions quite closely parallel the estimating equations considered in Section II for mixtures of two Poisson distributions. Consider a compound exponential distribution with probability density function

$$f(x) = \alpha(1/\mu)e^{-x/\mu} + (1 - \alpha)(1/\lambda)e^{-x/\lambda}. \quad \begin{cases} x \geqq 0 \\ \mu > \lambda > 0 \\ 0 \leqq \alpha \leqq 1 \end{cases} \tag{21}$$

The nonessential restriction that $\mu > \lambda$ is imposed as a matter of convenience and without any loss of generality.

The k^{th} noncentral moment of x is

$$m'_k = \int_0^\infty x^k \; f(x) \; dx = k! \; [\alpha\mu^k + (1 - \alpha) \; \lambda^k]. \tag{22}$$

Accordingly, the first three noncentral moments are

$$\left.\begin{aligned} m'_1 &= \alpha\mu + (1 - \alpha) \; \lambda \\ m'_2 &= 2[\alpha\mu^2 + (1 - \alpha) \; \lambda^2] \\ m'_3 &= 6[\alpha\mu^3 + (1 - \alpha) \; \lambda^3] \end{aligned}\right\} . \tag{23}$$

2. Three-Moment Estimators

When the first three noncentral sample moments, designated ν_1', ν_2' and ν_3', respectively, with $\nu_1' = \bar{x}$, are equated to the theoretical moments of (23), we obtain the estimating equations

$$\left.\begin{aligned} \bar{x} - \lambda &= \alpha(\mu - \lambda) \\ \frac{\nu_2'}{2} - \lambda^2 &= \alpha(\mu^2 - \lambda^2) \\ \frac{\nu_3'}{6} - \lambda^3 &= \alpha(\mu^3 - \lambda^3) \end{aligned}\right\} . \tag{24}$$

These equations differ from the corresponding equations for mixed Poisson distributions only in that $\nu_2'/2$ and $\nu_3'/6$ have replaced the factorial moments $\nu_{[2]}$ and $\nu_{[3]}$ of the mixed Poisson distribution.

On eliminating α between the first and second and between the first and third equations of (24), we simplify to obtain

$$\left.\begin{aligned} \bar{x}\theta - \Gamma &= \frac{\nu_2'}{2} \\ \bar{x}(\theta^2 - \Gamma) - \Gamma\theta &= \frac{\nu_3'}{6} \end{aligned}\right\} , \tag{25}$$

which are completely analogous to the last two equations of (2) in the case of mixed Poisson distributions. Here, as in the Poisson case, θ and Γ are defined by equation (3). Accordingly, on solving the two equations of (25) simultaneously, we have as estimators of θ and Γ

$$\left.\begin{aligned} \theta^* &= \frac{\dfrac{\nu_3'}{6} - \bar{x}\dfrac{\nu_2'}{2}}{\dfrac{\nu_2'}{2} - \bar{x}^2} \\ \Gamma^* &= \bar{x}\theta^* - \frac{\nu_2'}{2} \end{aligned}\right\} , \tag{26}$$

which are analogous to equation (5) for the mixed Poisson distribution.

Finally, with θ^* and Γ^* determined from (26), μ^* and λ^* follow from equation (6) as in the Poisson case, and α^* follows from the first equation of (24) as

$$\alpha^* = \frac{\bar{x} - \lambda^*}{\mu^* - \lambda^*} . \tag{27}$$

3. Estimation With Some Parameters Specified

a. α Known

We need only replace $\nu_{[2]}$ with $\nu_2'/2$ and the quadratic equation of (12) becomes, for the present case,

$$\lambda^2 - 2\bar{x}\lambda + \frac{\bar{x}^2 - \alpha \frac{\nu_2'}{2}}{1 - \alpha} = 0. \tag{28}$$

Accordingly,

$$\left.\begin{aligned} \lambda^* &= \bar{x} - \sqrt{\frac{\alpha(\frac{\nu_2'}{2} - \bar{x}^2)}{1 - \alpha}} \\ \mu^* &= \frac{\bar{x} - \lambda^*(1 - \alpha)}{\alpha} \end{aligned}\right\} . \tag{29}$$

b. α and μ Known

In this case, the estimator for λ follows from the first equation of (24) as

$$\lambda^* = \frac{\bar{x} - \alpha\mu}{1 - \alpha} , \tag{30}$$

which is identical with the corresponding estimator, equation (15), in the Poisson case.

c. α and λ Known

In this case, it follows from the first equation of (24) that

$$\mu^* = \frac{\bar{x} - (1 - \alpha)\lambda}{\alpha} . \tag{31}$$

d. λ Known

In this case, we need only replace $\nu_{[2]}$ in equation (19) with $\nu_2'/2$ and, accordingly,

$$\left.\begin{aligned} \mu^* &= \frac{\frac{\nu_2'}{2} - \bar{x}\lambda}{\bar{x} - \lambda} \\ \alpha^* &= \frac{\bar{x} - \lambda^*}{\mu - \lambda^*} \end{aligned}\right\} . \tag{33}$$

IV. COMPUTATIONAL PROCEDURES

The solution of the transcendental estimating equation (8) from Section II provides an interesting illustration of iterative numerical computational techniques described by Whittaker and Robinson (loc. cit.). To facilitate solution of equation (8), the denominator of the left side is interchanged with the numerator of the right side, and the resulting equation becomes

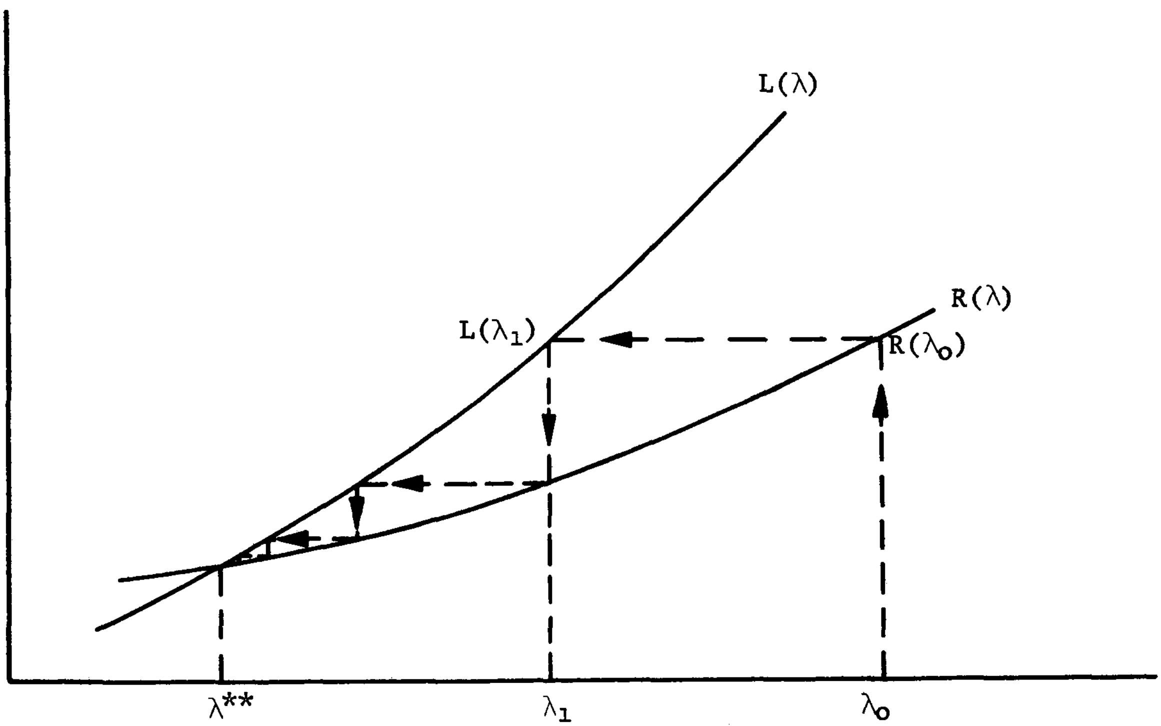
$$\frac{\bar{x} - \lambda}{n_0/n - e^{-\lambda}} = \frac{G(\lambda) - \lambda}{e^{-G(\lambda)} - e^{-\lambda}} , \tag{34}$$

where $G(\lambda)$ remains as given by equation (9).

Equation (34) might be condensed to the form $L(\lambda) = R(\lambda)$ where

$$L(\lambda) = \frac{\bar{x} - \lambda}{n_o/n - e^{-\lambda}} \quad \text{and} \quad R(\lambda) = \frac{G(\lambda) - \lambda}{e^{-G(\lambda)} - e^{-\lambda}} \,. \tag{35}$$

The two functions $L(\lambda)$ and $R(\lambda)$ are essentially as plotted below.



We begin with an initial approximation λ_o and iterate toward the value λ^{**} as described by Whittaker and Robinson [4, pp. 81-83]. The three-moment estimate of λ given by equation (6) of Section II provides a satisfactory value for λ_o. This initial approximation is substituted into the second equation of (35) to obtain R_o, which is merely an abbreviated notation for $R(\lambda_o)$. We then solve the equation

$$L(\lambda_1) = R_o \tag{36}$$

to obtain λ_1, the next approximation. This cycle is repeated as many times as necessary to attain the desired degree of accuracy. Equation (36) is itself a transcendental equation, though somewhat simpler in form than the original equation (34). It is amenable to solution by the Newton-Raphson method [4, pp. 84-86]. For the i^{th} cycle of iteration, the equation corresponding to (36) becomes

$$L(\lambda_i) = \frac{\bar{x} - \lambda_i}{n_o/n - e^{-\lambda_i}} = R_{i-1}, \tag{37}$$

which may be written as

$$f(\lambda_i) = 0,$$

where

$$f(\lambda_i) = \lambda_i - R_{i-1}\, e^{-\lambda_i} - C_{i-1} \tag{38}$$

and

$$C_{i-1} = (\bar{x} - R_{i-1}\, n_o/n).$$

Equation (37) may be readily solved using the Newton-Raphson method, where $\lambda_{i:r+1}$, the $(r + 1)^{st}$ iterant to λ_i, is given by

$$\lambda_{i:r+1} = \lambda_{i:r} - \frac{f(\lambda_{i:r})}{f'(\lambda_{i:r})}.$$

The first derivative of $f(\lambda_i)$ follows from equation (38) as

$$f'(\lambda_i) = 1 + R_{i-1}\, e^{-\lambda_i}.$$

Accordingly,

$$\lambda_{i:r+1} = \lambda_{i:r} - \left[\frac{\lambda_{i:r} - R_{i-1}\, e^{-\lambda_{i:r}} - C_{i-1}}{1 + R_{i-1}\, e^{-\lambda_i}}\right]. \tag{39}$$

As an initial approximation $\lambda_{i:0}$ to λ_i, it will usually be satisfactory to let $\lambda_{i:0} = \lambda_{i-1}$. The Newton-Raphson iterative technique is continued through as many cycles as necessary to attain the desired accuracy in λ_i. More specifically, this procedure is terminated at the end of the r^{th} cycle, the first cycle for which

$$|L_{i:r} - R_{i-1}| < \delta_1 ,$$

where δ_1 specifies the maximum permissible absolute value deviation. With λ_i thus determined, we calculate R_i, set up the new equation

$$L(\lambda_{i+1}) = R_i ,$$

and continue the primary routine through k cycles. The k^{th} cycle is the first for which

$$|L_k - R_k| < \delta_2, \tag{40}$$

where δ_2 specifies the maximum allowable absolute value deviation. The required estimate of λ is then

$$\lambda^{**} = \lambda_k.$$

V. ILLUSTRATIVE EXAMPLES

1. Mixed Poisson Distribution

To illustrate the application of his three-moment estimators, Rider [3] chose an example constructed by mixing equal proportions of two Poisson distributions with $\mu = 1.5$ and $\lambda = 0.5$, respectively. These data are as follows:

x	0	1	2	3	4	5	6	7
n_x	830	638	327	137	49	15	3	1

In summary, $n = 2{,}000$, $n_o = 830$, $\bar{x} = 0.9995$, $\nu_{[2]} = 1.243$ and $\nu_{[3]} = 1.734$. Direct substitution of these values into equations (5) and (6) yields the three-moment estimates

$$\mu^* = 1.4766563,$$

$$\lambda^* = 0.47765894,$$

$$\alpha^* = 0.52236479.$$

The above results differ slightly from those given by Rider due, apparently, to small round-off errors in his calculations.

Estimates based on the first two sample moments and the sample zero-frequency, calculated by a computer program of the routine described in Section IV, are

$$\mu^{**} = 1.4936,$$

$$\lambda^{**} = 0.4956,$$

$$\alpha^{**} = 0.5049.$$

These estimates are in much closer agreement with the actual population parameters $\mu = 1.5$, $\lambda = 0.5$, and $\alpha = 0.5$ than the three-moment estimates. Investigations are continuing with regard to the relative efficiency of the three-moment and the two-moment plus zero-frequency estimates; but at least in the present instance, where a large proportion of the population is in the zero class, the two-moment plus zero-frequency estimates seem to be preferred.

2. Mixed Exponential Distribution

To illustrate the application of estimators derived in this case, a sample of 2000 observations was selected from a mixed population constructed by combining two exponential distributions with $\mu = 2$, $\lambda = 1$, and $\alpha = 0.4$. Data for the sample selected are summarized as follows: $n = 2{,}000$, $\bar{x} = 1.42$, $\nu_2' = 4.38$, and $\nu_3' = 21.6$.

Direct substitution of these data into equations (26), (6), and (27) yields as three-moment estimates:

$$\mu^* = 1.85,$$

$$\lambda^* = 1.02,$$

$$\alpha^* = 0.48.$$

APPENDIX

FIND - A Computer Program

By

Frank C. Clark

FIND is a Fortran IV computer program which calculates estimates for the parameters α, μ, and λ of a compound (mixed) Poisson distribution. These estimates are calculated from (1) the first three sample moments and (2) the first two sample moments and the sample zero-frequency.

In finding λ for the second case, the following equation is solved:

$$\frac{\bar{x} - \lambda}{G(\lambda) - \lambda} = \frac{n_o/n - e^{-\lambda}}{e^{-G(\lambda)} - e^{-\lambda}},$$

where

$$G(\lambda) = \frac{\nu_{[2]} - \bar{x}\lambda}{\bar{x} - \lambda},$$

and $\nu_{[2]}$, $\bar{x}$, and n_o/n are known constants. FIND makes use of the Newton-Raphson and geometrical iteration methods [4] in solving the equation.

FIND requires, for each data sample, input values for $\bar{x}$, n_o/n, $\nu_{[2]}$, and $\nu_{[3]}$, punched on a single card. Iteration continues through $k < 500$ cycles until the absolute error of equation (40) is less than 0.00001, i.e., until

$$|L_k - R_k| < 0.00001.$$

If this criteria is not met when k = 500, the message "completed 500 iterations with no success" is given and the program stops. Should greater accuracy be required in the estimate of λ, appropriate change should be made in the source program card "TOL = .00 ... ."

FIND prints out the following:

1. Values of the index, i.

2. Values of λ in the Newton-Raphson iteration.

3. Values of

   $$\text{ERROR} = \text{TEST 1} - \text{TOL},$$

   where

   $$\text{TEST 1} = |L_{i:r} - R_{i-1}|.$$

4. α, μ, and λ based on the first three sample moments. (This value of λ is used as the first approximation in the Newton-Raphson process.)

5. α, μ, and λ based on the first two sample moments and the sample zero-frequency.

FIND (FORTRAN IV)

```
C.....ESTIMATION IN MIXTURES OF TWO POISSON DISTRIBUTIONS
      DIMENSION   LAM(4000)
      REAL MU,NUE2,NUE3,NON,LAM,L,LAMBDA,MU1
    1 READ(5,2)XBAR,NON,NUE2,NUE3
    2 FORMAT(4F10.5)
      THET = (NUE3-XBAR*NUE2)/(NUE2-(XBAR**2))
      CLAM = XBAR*THET-NUE2
      MU = (THET+SQRT(THET**2-4.0*CLAM))/2.0
      I=1
      LAM(I)= (THET-SQRT(THET**2-4.0*CLAM))/2.0
      N=0
      ALPHA1= (XBAR-LAM(I))/(MU-LAM(I))
      K=0
      G    = (NUE2-XBAR*LAM(I))/(XBAR-LAM(I))
      N=N+1
      R   = (G-LAM(I))/(EXP(-G)-EXP(-LAM(I)))
    9 C = (XBAR-NON*R)
   10 K=K+1
      LAM(I+1) = LAM(I)-((LAM(I)-R*EXP(-LAM(I))-C)/(1.0+R*EXP(-LAM(I))))
      L = (XBAR-LAM(I+1))/(NON-EXP(-LAM(I+1)))
      TOL = .00001
      TEST1 = ABS(L-R)
C     ......
   60 FORMAT(1H ,I5,5X,E15.8,E15.8)
      ERROR = TEST1 - TOL
      WRITE(6,60)I,LAM(I),ERROR
      IF (TEST1-TOL)20,15,15
   23 I=I+1
      GO TO 10
   20 G = (NUE2-XBAR*LAM(I+1))/(XBAR-LAM(I+1))
      R = (G-LAM(I+1))/(EXP(-G)-EXP(-LAM(I+1)))
      TEST2 = ABS(L-R)
      IF (TEST2-TOL)30,25,25
   24 I=I+1
      K=0
      GO TO 9
C     ......
   15 IF(500-K)22,22,23
   25 IF(500-N)22,22,24
   22 WRITE(6,28)
   28 FORMAT(42H1COMPLETED 500  ITERATIONS WITH NO SUCCESS)
      GO TO 100
   30 MU1      = (NUE2-XBAR*LAM(I+1))/(XBAR-LAM(I+1))
      LAMBDA   =  LAM(I+1)
      ALPHA2   =  (XBAR-LAM(I+1))/(MU1    -LAM(I+1))
C     ......
      WRITE(6,50)
   50 FORMAT(39H1ESTIMATES BASED ON FIRST THREE MOMENTS)
      WRITE(6,51)MU,LAM(1),ALPHA1
   51 FORMAT(10H0     MU = E15.8,10H LAMBDA = E15.8,9H ALPHA = E15.8)
      WRITE(6,52)
   52 FORMAT(74H0ESTIMATES BASED ON FIRST TWO SAMPLE MOMENTS AND THE ZER
     1O SAMPLE FREQUENCY)
      WRITE(6,53)MU1,LAMBDA,ALPHA2
   53 FORMAT(11H0      MU = E15.8,10H LAMBDA = E15.8,10H ALPHA =  E15.8)
      GO TO 1
  100 STOP
      END
```

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MIXTURES OF EXPONENTIAL DISTRIBUTIONS

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The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

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Director, Aero-Astrodynamics Laboratory

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